

# COUNTING DYONS IN $N = 4$ STRING THEORY

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## ABSTRACT

We present a microscopic index formula for the degeneracy of dyons in four-dimensional  $N = 4$  string theory. This counting formula is manifestly symmetric under the duality group, and its asymptotic growth reproduces the macroscopic Bekenstein-Hawking entropy. We give a derivation of this result in terms of the type II five-brane compactified on  $K3$ , by assuming that its fluctuations are described by a closed string theory on its world-volume. We find that the degeneracies are given in terms of the denominator of a generalized super Kac-Moody algebra. We also discuss the correspondence of this result with the counting of D-brane states.

## 1. INTRODUCTION

In this paper we will study the dyonic spectrum of  $N = 4$  string theory in 4 dimensions. This theory has two perturbative formulations: one in terms of the toroidally compactified heterotic string, and a dual formulation in terms of type II strings compactified on  $K3 \times T^2$ . To describe the set of charged states we will take the point of view of the heterotic string, so that we have 28 electric charges  $q_e$  and 28 magnetic charges  $q_m$  which both lie on the  $\Gamma^{22,6}$  lattice. This theory is conjectured [1, 2, 3] to have as its exact duality group

$$SL(2, \mathbf{Z}) \times SO(22, 6, \mathbf{Z}) \quad (1.1)$$

with  $SL(2, \mathbf{Z})$  being the electric-magnetic duality symmetry. The perturbative heterotic string states carry only electric charge, so that states with magnetic charge arise necessarily non-perturbatively as solitons.

Most accessible to computations is the spectrum of BPS states. These states are annihilated by some subset of the 16 supersymmetry charges and therefore many of their properties are protected in perturbation theory. The purely electric states are special in the sense that they preserve 1/2 of the supercharges and their degeneracies are easily determined, since they simply correspond to the heterotic string states that are in the right-moving ground state. Hence a BPS heterotic string state is described by specifying the 28 charges  $q_e \in \Gamma^{22,6}$  together with the occupation numbers  $N_l^I$  ( $I = 1, \dots, 24$ ,  $l > 0$ ), subject to the level-matching condition

$$\frac{1}{2}q_e^2 + \sum_{l,I} l N_l^I = 1. \quad (1.2)$$

Here the subscript  $l$  to  $N_l^I$  denotes the world-sheet oscillation number of the coordinate field  $X^I$ , and  $q_e^2$  is defined using the  $SO(22, 6)$  invariant inner product on the  $\Gamma^{22,6}$  lattice. The number of such states is given by\*

$$d(q_e) = \oint d\sigma \frac{e^{i\pi\sigma q_e^2}}{\eta(\sigma)^{24}} \quad (1.3)$$

where the ‘contour’ integral over  $\sigma$  is from 0 to 1 and  $\eta(\sigma)$  is the Dedekind  $\eta$ -function.

The conjectured electric-magnetic  $SL(2, \mathbf{Z})$ -duality predicts that there should also exist a solitonic version of the heterotic string that carries pure magnetic charge  $q_m \in \Gamma^{22,6}$ , and thus that a similar formula counts the pure magnetically charged 1/2 BPS states. The generic dyonic states, however, preserve only 1/4 of the supersymmetries and are more

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\*Here and in the subsequent we omit a factor of 16, and therefore count the number of 1/2 BPS multiplets rather than individual states.

mysterious. In the following we will give a concrete proposal for the exact degeneracy of these dyonic BPS states. To be more precise, we will present a formula for the number of bosonic minus the number of fermionic BPS-multiplets for a given electric and magnetic charge. The basic idea is the following. We like to think of the dyonic states as some kind of bound state of an electric heterotic string with a dual magnetic heterotic string. The formula that counts these states, should therefore be a suitable generalization of (1.3), that will contain (1.3) as a special subcase. In addition it has to satisfy other non-trivial consistency checks, such as invariance under the full duality group, while it also has to reproduce the correct asymptotic growth as predicted from the macroscopic entropy formulas of extremal dyonic black holes [4, 5].

After presenting our formula and illustrating some of its features, we will show that the complete duality invariant dyon spectrum has a unified and natural interpretation in terms of strings on the five-brane. We will also discuss the correspondence with the counting of BPS-states that follows from a D-brane description [6, 7, 8, 9].

## 2. THE DEGENERACY FORMULA

To write the proposed dyonic index formula, it will be convenient to combine the electric and magnetic charge vectors  $q_e$  and  $q_m$  into a vector as follows

$$\mathbf{q} = \begin{pmatrix} q_m \\ q_e \end{pmatrix} \quad (2.1)$$

Accordingly we introduce the  $2 \times 2$  matrix

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} \quad (2.2)$$

generalizing the single modulus  $\sigma$  in (1.3). The degeneracy formula will then take the following form

$$d(q_e, q_m) = \oint d\Omega \frac{e^{i\pi \mathbf{q} \cdot \Omega \cdot \mathbf{q}}}{\Phi(\Omega)}. \quad (2.3)$$

The integrals over the moduli parameters  $\sigma, \rho$  and  $v$  again all run over the domain from 0 to 1, and impose level matching conditions analogous to (1.2).

In the above suggestive form, it is natural to identify the matrix  $\Omega$  with the period matrix of a genus two Riemann surface. In the result that we will obtain, the denominator  $\Phi(\Omega)$  indeed will turn out to be a genus two modular form. More precisely, it is the unique automorphic form of weight 10 of the modular group  $Sp(2, \mathbf{Z})$  (represented by

$4 \times 4$  matrices) and can be expressed as the squared product of all genus 2 theta functions with even spin structure

$$\Phi(\Omega) = 2^{-12} \prod_{\alpha=\text{even}} \theta[\alpha](\Omega)^2. \quad (2.4)$$

$\Phi(\Omega)$  also happens to be equal to the denominator in the Weyl-Kac-Borcherds character formula for a generalized super Kac-Moody algebra, and as such it is a special case of the automorphic forms constructed in [10]. The  $SL(2, \mathbf{Z})$  duality transformations are identified with the subgroup of  $Sp(2, \mathbf{Z})$  that leave the genus two modular form  $\Phi(\Omega)$  invariant, and thus the presented degeneracies are manifestly duality symmetric. In addition, the formula satisfies a number of other non-trivial consistency checks, that we will now discuss. Subsequently, we will turn to the derivation and more detailed description of this result.

To make our formula somewhat more explicit, we can expand the integrand as a formal Fourier series expansion

$$\frac{1}{\Phi(\Omega)} = \sum_{k,l,m} D(k,l,m) e^{-2\pi i(k\rho + l\sigma + mv)} \quad (2.5)$$

with  $k, l, m$  integers. The coefficients  $D(k, l, m)$  are all integers, that via (2.3) give us the degeneracy for a given electric and magnetic charge as

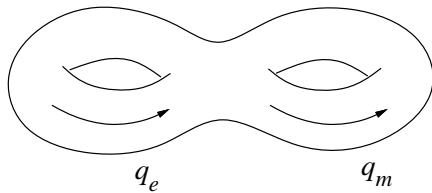
$$d(q_m, q_e) = D(\tfrac{1}{2}q_m^2, \tfrac{1}{2}q_e^2, q_e \cdot q_m) \quad (2.6)$$

Note that the normalization of (2.4) is chosen such that  $d(0, 0) = 1$ .

### *Correspondence with the heterotic string*

As we will explain in the following, the parameter  $v$  couples to the helicity  $m$  of the dyonic states. The integral over  $v$  thus projects on dyons with helicity equal to zero. However, instead of integrating over  $v$  we can also put it to fixed value, like  $v = 0$ . The resulting formula will then correspond to a helicity trace  $(-1)^m$ , which will project out all 1/4 BPS representations, except for the purely electric or magnetic 1/2 BPS representations. Hence, for  $v \rightarrow 0$  the integrand in formula (2.3) should match with the degeneracy formula (1.3) for the heterotic string BPS spectrum.

To make this correspondence explicit, we note that the function  $1/\Phi(\Omega)$  is in fact identical to the chiral half of the bosonic string partition function (or rather, that of 24 massless scalars) on the corresponding genus two curve (see [11]). This means in particular that, if we factorize the genus two surface parametrized by  $\Omega$  into two separate genus one surfaces with moduli  $\rho$  and  $\sigma$ , we indeed recover the separate 1/2 BPS partition sums. In



**Fig. 1:** *The dyon counting can be represented in terms of a genus two partition sum. The electric and magnetic charges of the dyon correspond to the loop momenta through the two handles of the Riemann surface.*

this analogy the electric and magnetic charges correspond to the loop momenta through the two handles, as in *fig. 1*.

Concretely, this factorization corresponds to taking the limit  $v \rightarrow 0$ , under which the function  $1/\Phi(\Omega)$  reduces to

$$\frac{e^{i\pi\mathbf{q}\cdot\Omega\cdot\mathbf{q}}}{\Phi(\Omega)} \longrightarrow \frac{1}{v^2} \frac{e^{i\pi\rho q_m^2}}{\eta(\rho)^{24}} \frac{e^{i\pi\sigma q_e^2}}{\eta(\sigma)^{24}} \quad (2.7)$$

The diverging factor  $1/v^2$  corresponds to the ‘tachyon pole,’ and reflects the emergence of two non-compact translational zero-modes for  $v = 0$ . In the other factors we recover the separate electric and magnetic heterotic string contributions. The above correspondence is a necessary boundary condition on any (candidate) dyonic degeneracy formula.

#### *Asymptotic growth and black hole entropy*

As a further non-trivial boundary condition, the formula (2.3) also matches the asymptotic behavior for large charges that one obtains by comparing with the results for the macroscopic Bekenstein-Hawking entropy of extremal 4-dimensional black holes. This correspondence predicts that the statistical entropy of states with charge  $(q_e, q_m)$  is given by [5]

$$S = \pi \sqrt{q_e^2 q_m^2 - (q_e \cdot q_m)^2}. \quad (2.8)$$

The asymptotic growth of the degeneracies  $d(q_e, q_m)$  indeed agrees with this predicted entropy, as it can be shown that in the large charge limit

$$d(q_e, q_m) \sim e^{\pi \sqrt{q_e^2 q_m^2 - (q_e \cdot q_m)^2}}. \quad (2.9)$$

A formal derivation of this result is outlined in the Appendix, and makes use of the zeroes of the function  $\Phi$ . These zeroes lead to poles in the integrand in (2.3). In our

case the leading contribution to the integral comes from the pole at  $\rho\sigma - v^2 + v = 0$  (which is an appropriate  $Sp(2, \mathbf{Z})$ -transform of the above described ‘tachyon pole’). After evaluating the residue at this pole one can perform the remaining integration by means of a saddle-point approximation, and this immediately reproduces the above formula for the asymptotic growth of  $d(q_e, q_m)$ . We will give an alternative explanation of this result at the end of the next section, after we have reinterpreted the degeneracy formula (2.3) in terms of strings living on a five-brane.

### 3. DERIVATION FROM THE QUANTUM FIVEBRANE

It is known that the heterotic string arises as a soliton in type II string theory that can be described as a five-brane wrapped around the internal  $K3$  manifold [12].  $SL(2, \mathbf{Z})$  symmetry in the type II formulation is a consequence of  $T$ -duality, and thus string-string duality implies that both the electric and magnetic heterotic string can be considered as five-branes wrapped around  $K3$ . Based on the above the degeneracy formula, we will argue in the following that this correspondence can be naturally extended to all dyonic states.

#### *Heterotic string from the five-brane*

The five-brane we will consider arises as a soliton in the 10-dimensional type II effective string theory [13]. The long-wavelength fields on its world-volume are given by an  $N = (2,0)$  supermultiplet consisting of a tensor with self-dual 3-index field strength  $T_3$ , 4 chiral fermions and 5 scalar fields. These fields represent the zero modes of the 10-dimensional massless fields on the five-brane background, which, via the standard adiabatic argument, are allowed to vary along the five-brane world-volume.

The relation of the five-brane with the heterotic string can be derived by dimensionally reducing the six-dimensional chiral field theory on the world-volume to two dimensions on  $K3$ . In this way the self-dual tensor field reduces to 19 left-moving and 3 right-moving scalars. Together with the additional five scalars and fermions, this gives exactly the world-sheet fields of the heterotic string. Moreover, the string momentum lattice  $\Gamma^{20,4}$  corresponds to the cohomology lattice of  $K3$ . In particular, the fluxes of the self-dual tensor field  $T_3$  are identified with the electric charges

$$q_e^A = \int_{S^1 \times \Sigma^A} T_3 \quad (3.1)$$

where  $S^1$  is one of the cycles of  $T^2$  and  $\Sigma^A$  is a basis of two-cycles in  $K3$ .

In our setup, we would in fact like to take this correspondence one step further, by replacing the standard world-sheet theory of the heterotic string by the 5+1-dimensional world-volume theory of the five-brane wrapped around  $K3$ . Concretely this means that in counting the states of this generalized heterotic string, we wish to take seriously the possible fluctuations of the five-brane in the internal  $K3$ . At first sight this would appear to imply a very drastic extension of the world-sheet degrees of freedom, but it turns out that for purely electrically charged BPS states nothing has changed: when one imposes the BPS conditions one automatically finds that the allowed fluctuations of the five-brane are the usual left-moving heterotic string oscillations [14]. We will outline this calculation in a moment, after we have explained our general set-up.

### *Idea of the computation*

We would like to extend the above description to include dyonic states. As we will argue in the following, these dyonic states will naturally arise as 1/4 BPS configurations of the five-brane, provided we make one important new assumption. Namely, we will assume that the world volume description of the five-brane is not described by a field theory, but rather by an appropriate closed string theory. This world-volume string theory must satisfy the condition that the usual massless collective fields should correspond to the ground states of this string, in direct analogy with the standard relation between the space-time effective field theory and the ground states of the type II string. In this sense, we wish to think about the strings on the five-brane as essentially the original type II string, but restricted to the world-volume. The effective field theory of the string ground states is sufficient to describe the long-wavelength fluctuations of the five-brane, but at short distance we will also take into account the other string modes, including winding configurations and excitations.

Our aim is eventually to determine the degeneracy of BPS states. We will do this by considering the partition function of the 5+1-dimensional world-volume string theory with target space  $K3 \times T^2$ .<sup>\*</sup> In a light-cone gauge in which the transversal space is taken to be the  $K3$  manifold, the worldsheet theory on this string is a  $N = (4, 4)$  superconformal field theory with  $SU(2)_L \times SU(2)_R$  current algebra [15, 14]. The eight supercharges on the world-sheet describe the unbroken part of the space-time supersymmetry. (The broken space-time supersymmetries give rise to zero-modes on the five-brane.)

The five-brane BPS partition function  $\mathcal{Z}$  essentially counts the number of BPS states of this second quantized string theory. In computing this quantity, we will neglect string interactions. The justification for this should come from the fact that the BPS-condition in combination with the large number of supersymmetries severely restricts the possible

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<sup>\*</sup>This calculation is the direct analog of the conventional counting of string states by means of a conformal field theory partition function on a world-sheet of topology  $T^2$ .

interactions. This expectation will indeed find concrete support in our analysis, as it will turn out that for the relevant string partition sums on the target space  $K3 \times T^2$ , there are no higher loop correction to the free energy. Hence the fivebrane BPS partition sum can be computed by taking the exponent of the relevant one-loop free energy of the string on the world-volume, provided one also adds an appropriate zero-mode contribution

$$\mathcal{Z} = \exp[\mathcal{F}_{1-loop}^{string} + \mathcal{F}_{0-modes}]. \quad (3.2)$$

The concrete form of this zero-mode contribution follows from considering the free energy of the fluxes of the low-energy massless field theory on the five-brane world-volume. The partition sum  $\mathcal{Z}$  will depend in particular on the moduli that parametrize the shape and size of world-volume manifold.

To obtain the number of BPS states for given electric and magnetic charge, one finally needs to identify the corresponding contribution to the partition sum. In our formalism, this identification is automatically implemented via the integral over an appropriate set of world-volume moduli, which act as Lagrange multipliers that impose ‘level-matching conditions’ analogous to the  $L_0 - \bar{L}_0 = 0$  condition in string theory. We will now explain and motivate this procedure in more detail for the 1/2 BPS states, in which case we should recover the before-mentioned classical correspondence with the electric and magnetic heterotic string.

#### *Electric and magnetic fivebrane states*

To count the states that respect half of the supersymmetries we have to restrict to the string ground states. This turns the 6-dimensional string into a topological string for which we can simply compute its one-loop free energy  $\mathcal{F}$ . The one-loop computation is well-known [16]. There are 24 ground states on  $K3$ , corresponding to the harmonic forms. We further add the momenta and winding numbers in the  $T^2$  direction, which take value in the lattice  $\Gamma^{2,2}$  parametrised by the moduli  $\rho, \sigma$ . Here  $\rho$  parametrize the size and  $B$ -field on the two-torus and  $\sigma$  parametrizes the complex structure. We then compute [16]

$$\begin{aligned} \mathcal{F} &= \frac{1}{2} \int \frac{d^2\tau}{\tau_2} \sum_{(p_L, p_R) \in \Gamma^{2,2}} e^{i\pi(\tau p_L^2 - \bar{\tau} p_R^2)} \cdot 24 \\ &= 24 \log(\rho_2^{1/2} |\eta(\rho)|^2) + 24 \log(\sigma_2^{1/2} |\eta(\sigma)|^2) + cns. \end{aligned} \quad (3.3)$$

Here the first term represents the contribution of the winding modes of the strings that couple to the parameter  $\rho$  and the second term the contribution of the momentum modes that couple to the parameter  $\sigma$ . The occurrence of these two terms reflects a  $\mathbf{Z}_2$  symmetry,



which is an  $R \rightarrow 1/R$  symmetry applied to the world-volume of the five-brane. This transformation interchanges momentum and winding modes. So, even at the level of the string ground states, we encounter a pure stringy effect.

To obtain the 1/2 BPS partition function there are two more steps to be taken. First, we have to take only the holomorphic part of the above expression in  $\rho$  and  $\sigma$ , and secondly, we need to include the zero-mode contributions. Specifically, we must replace the factor  $\sigma_2$  in  $\mathcal{Z} = e^{\mathcal{F}}$  by the appropriate theta-function representing the summation of the fluxes over the lattice  $\Gamma^{22,6}$ , see [17]. Combined with the momentum contribution  $1/\eta(\sigma)^{24}$  this then gives the left-moving partition function of the heterotic string with modular parameter  $\sigma$ . This is in accord with the expected one-to-one correspondence between 1/2 BPS states of the five-brane and those of the heterotic string.

To restore the  $\mathbf{Z}_2$  symmetry between  $\rho$  and  $\sigma$ , we also need to replace the powers of  $\rho_2$  by a lattice sum with a new set of charges. It seems natural to identify these new charges with the magnetic charges  $q_m \in \Gamma^{22,6}$ . This identification finds support from the heterotic-type II string duality, as follows. The electric-magnetic duality map that interchanges  $q_e$  and  $q_m$  acts on the type IIA side as a  $T$ -duality on the  $T^2$ . Since the five-brane we consider is wrapped around  $K3$  and a one-cycle in  $T^2$ , this transformation will act non-trivially on its world-volume via an  $R \rightarrow 1/R$  transformation on the  $S^1$ . As we have just discussed, this is a symmetry of the world-volume string theory that interchanges the world-volume parameter  $\rho$  and the parameter  $\sigma$ .

Hence, we conclude that by representing the world-volume degrees of freedom on the five-brane as strings, we are able to describe the electrically charged heterotic string and the magnetically charged heterotic string both in terms of one single fivebrane.

### *Dyonic fivebrane states*

We now wish to consider five-brane states that carry both electric and magnetic charge. These dyonic states necessarily break 3/4 of the 16 supercharges. Concretely this means that the dyonic BPS five-branes carry more degrees of freedom than the purely electric or magnetic configurations.

As already noted, the 8 unbroken supercharges of the five-brane are realized on the world-sheet of the string as the 4 left-moving and 4 right-moving generators of an  $N = (4, 4)$  superconformal algebra. So to keep 4 unbroken charges we must restrict the string states to be in the ground state of either the right-moving or the left-moving sector. For definiteness, we will choose the right-moving one. The index that counts these string states is given by the elliptic genus of  $K3$  [18], which is defined as the weighted trace in the RR-sector of the superconformal sigma-model

$$\chi_{\tau,z}(K3) = \text{Tr} \left( (-1)^{F_L + F_R} e^{2\pi i(\tau(L_0 - \frac{c}{24}) + zF_L)} \right) \quad (3.4)$$

with  $c = 6$  for  $K3$ . The fermion numbers  $F_L$  and  $F_R$  can be identified with the zero-modes of the left-moving and right-moving  $U(1) \subset SU(2)$  current algebras.

As we restrict to right-moving ground states,  $F_R$  only takes the values  $0, \pm 1$ , while  $F_L$  can a priori take all integer values and gives us a conserved quantum number  $m$ . Since the corresponding  $SU(2)$ -symmetry is part of the space-time Lorentz symmetry, this conserved quantum number can be interpreted as the helicity in space-time, *cf* [19, 20]. In the following we will treat it on equal footing with the momentum and winding numbers of the string around  $T^2$ . This turns the Narain lattice  $\Gamma^{2,2}$  of the two-torus into the lattice  $\Gamma^{3,2}$ . Moreover, it allows us to extend the set of world-brane parameters  $\rho, \sigma$  by another complex parameter  $v$  that couples to the helicity quantum number  $m$ . Technically,  $v$  has an interpretation as a Wilson loop on the world-volume that parametrizes the  $SU(2)_L$  bundle over  $T^2$ . Together,  $\rho, \sigma$  and  $v$  parametrize the lattice  $\Gamma^{3,2}$ , and transform in a vector representation of the  $SL(2, \mathbf{Z})$  subgroup of the corresponding  $SO(3, 2; \mathbf{Z})$   $T$ -duality group. (See Appendix.)

We now need to compute the following one-loop integral

$$\mathcal{F} = \frac{1}{2} \int \frac{d^2\tau}{\tau_2} \sum_{\substack{(p_L, p_R) \in \Gamma^{3,2} \\ n \in 4\mathbf{Z} - \epsilon}} e^{i\pi(\tau p_L^2 - \bar{\tau} p_R^2)} \cdot c(n) e^{i\pi\tau n/2} \quad (3.5)$$

where  $\epsilon = 0, 1$  depending on whether the helicity quantum number  $m$  is even or odd, and the coefficients  $c(n)$  are defined by the expansion of the  $K3$  elliptic genus as<sup>†</sup>

$$\chi_{\tau,z}(K3) = \sum_{h \geq 0, m \in \mathbf{Z}} c(4h - m^2) e^{2\pi i(h\tau + mz)} \quad (3.6)$$

It turns out that precisely the integral (3.5) has been computed recently by T. Kawai in the context of  $N = 2$  heterotic string compactifications [21] along the lines of [22]. The answer obtained in [21] reads

$$\mathcal{F} = -\log \left[ (\rho_2 \sigma_2 - v_2^2)^{10} |\Phi(\rho, \sigma, v)|^2 \right] + \text{const.} \quad (3.7)$$

where the holomorphic part has the following product representation

$$\Phi(\rho, \sigma, v) = e^{2\pi i(\rho + \sigma + v)} \prod_{(k,l,m) > 0} \left( 1 - e^{2\pi i(k\rho + l\sigma + mv)} \right)^{c(4kl - m^2)}. \quad (3.8)$$

Here  $(k, l, m) > 0$  means that  $k, l \geq 0$  and  $m \in \mathbf{Z}$ ,  $m < 0$  for  $k = l = 0$ . (Note further that  $c(n) = 0$  for  $n < -1$ ).

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<sup>†</sup>Up to normalization this is the unique weak Jacobi form of index 1 and weight 0.

Now, following the same logic as for the 1/2 BPS states, we define the partition sum for the dyonic 1/4 BPS states by  $\mathcal{Z} = e^{\mathcal{F}}$ , where again we have to include separately the zero-mode contributions and restrict to the holomorphic sector. So the denominator of the partition function is given by  $\Phi(\rho, \sigma, v)$  and represents the contribution from multi-string states that describe the fluctuations of the five-brane preserving the BPS condition. From the appearance of the factor  $\rho_2 \sigma_2 - v_2^2 = \det \text{Im } \Omega$  it is clear that the zero-modes must be included by a factor

$$\exp\left[i\pi(q_m^2 \rho + q_e^2 \sigma + 2q_e \cdot q_m v)\right] \quad (3.9)$$

Combining all ingredients, and continuing the parallel with the heterotic string discussion of the introduction, we finally arrive at the announced result for the (bosonic minus fermionic) degeneracy formula

$$d(q_e, q_m) = \oint d\rho d\sigma dv \frac{e^{i\pi(q_m^2 \rho + q_e^2 \sigma + 2q_e \cdot q_m v)}}{\Phi(\rho, \sigma, v)}. \quad (3.10)$$

As was also noted in [21], it is natural to combine the moduli  $\rho$ ,  $\sigma$  and  $v$  into a matrix  $\Omega$  as in (2.2). In the above form, the function  $\Phi(\rho, \sigma, v)$  is manifestly  $SL(2, \mathbf{Z})$  invariant, which suggests that it can be written as a modular form in  $\Omega$ . Indeed, a remarkable identity proved by Gritsenko and Nikulin [23] expresses the product representation (3.8) in terms of the product of even genus 2 theta-functions as in formula (2.4). The role of the automorphic form  $\Phi$  in string theory was first recognized in [24].

### *State counting*

The above single five-brane partition function has a concrete interpretation as a trace of the multi-string quantum states of a target space topology  $K3 \times S^1 \times \mathbf{R}$ . On such a target space a string carries three additive quantum numbers: a winding number  $k$ , a momentum  $l$  along the  $S^1$  and a helicity  $m$ . Together the three integers  $(k, l, m)$  form a vector on the lattice  $\Gamma^{2,1}$  with length-squared  $m^2 - 4kl$ , which is invariant under the symmetry group

$$SO(2, 1, \mathbf{Z}) \cong SL(2, \mathbf{Z}) \quad (3.11)$$

We claim that this  $SL(2, \mathbf{Z})$  group should be identified with the electric-magnetic duality group. Level-matching on the string tells us that the conformal dimension  $h$  in the  $K3$  sigma model must be set equal to  $h = kl$ . If we look at the elliptic genus formula we see that the number of single string states with given quantum numbers  $(k, l, m)$  is  $|c(4kl - m^2)|$ , and thus duality invariant. Such a string state represents a space-time

boson or fermion depending on the sign of  $c(4kl - m^2)$ . (The sign of  $c(n)$  is  $+1$  if  $n \equiv 0 \pmod{4}$  and  $-1$  if  $n \equiv -1 \pmod{4}$ , with the exception of  $c(-1) = 2$ .)

A state in the multi-string Hilbert space is characterized by a set of occupation numbers  $N_{k,l,m}^I$  with  $I = 1, \dots, |c(4kl - m^2)|$ . Hence, the degeneracy  $d(q_e, q_m)$  counts the number of independent multi-string states  $|\{N_{k,l,m}^I\}\rangle$  that satisfy the appropriate level-matching conditions implemented via the integral over the moduli  $\sigma$ ,  $\rho$  and  $v$ . These take the form

$$\begin{aligned} \frac{1}{2}q_m^2 + \sum_{k,l,m,I} k N_{k,l,m}^I &= 1 \\ \frac{1}{2}q_e^2 + \sum_{k,l,m,I} l N_{k,l,m}^I &= 1 \\ q_e \cdot q_m + \sum_{k,l,m,I} m N_{k,l,m}^I &= 1 \end{aligned} \tag{3.12}$$

The first two relations are obvious generalisations of the level-matching conditions of the heterotic string. In the case of only electric or magnetic charges they indeed reproduce the usual counting. The third line relates the sum of the helicities of the individual strings to the quantity  $q_e \cdot q_m$ . A suggestive interpretation of this constraint is that  $q_e \cdot q_m$  represents the contribution to the helicity as carried by the external electric-magnetic field of the dyon, when considered as made up from a separate electric and magnetic ‘point charge’. According to this interpretation, the third constraint in (3.13) essentially puts the total helicity of the dyonic state equal to zero.

#### 4. COMPARISON WITH D-BRANE COUNTING.

Here we would like to indicate how this representation of the BPS states is related to the more conventional construction in terms of D-branes [25, 26, 7, 9]. As we will argue, both the string representation as well as the level matching relations have a natural interpretation from this point of view.

##### *Strings and D-brane intersections*

Let us consider a five-brane wrapped around the  $K3$  compactification manifold, so that its world-volume has the space-like topology of  $K3 \times S^1$ . A pure five-brane only carries NS-NS charge. More general BPS-configurations that also carry R-R charge can be obtained by forming bound states with D-branes. For our purposes the most convenient description of these bound states is given in the context of type II B string theory. In this case we need to include 1-branes, 3-branes and 5-branes. When more of these D-branes

are taken to lie within the five-brane world volume, they will in general have intersections [26, 25]. The collection of BPS states that corresponds to a given charge configuration are obtained by quantizing the appropriate degrees of freedom of these intersections [7].

Since  $K3$  has only non-trivial homology cycles of even dimension, on the given topology of the five-brane of  $K3 \times S^1$ , all the type IIB D-branes will be wrapped around a corresponding element of  $H_*(K3)$  times the  $S^1$ . Hence to a given D-brane configuration we can associate a vector

$$q \in H_*(K3, \mathbf{Z}) \quad (4.1)$$

in the integral homology of  $K3$ . This vector becomes identified with the charge vector

$$q \in \Gamma^{20,4} \quad (4.2)$$

via the isomorphism of  $H_*(K3, \mathbf{Z}) \cong \Gamma^{20,4}$ , in which the norm on  $\Gamma^{20,4}$  is identified with the intersection form. A pair of 3-branes will typically intersect along a 1-dimensional string wound one or more times around the  $S^1$  direction, and similarly the 1-brane intersects with the 5-brane along a string around the  $S^1$ .

These strings formed by the D-brane intersections encode the internal degrees of freedom of the five-brane in a completely similar fashion as the string introduced in section 3. In particular, its transversal oscillations are also described by an appropriate  $N = 4$  superconformal field theory with target space  $K3$ , where the BPS-restriction again only allows for either left- or right-moving string oscillations. The spectrum of these quantized ‘intersection-strings’ will in particular include the excitations of the low-energy effective field theory on the five-brane world volume. The number of excited string states is again counted by means of the  $K3$  elliptic genus.

It is evident from the above description that the total string winding number is given by the intersection pairing of the homology class (4.1) on  $K3$  defined by the full D-brane configuration. Hence we see that the D-brane representation gives a clear geometrical interpretation of the matching condition

$$\frac{1}{2}q_m^2 + \sum_{k,l,m,I} k N_{k,l,m}^I = 1. \quad (4.3)$$

Here we identified the vector in (4.1) with the magnetic charge  $q_m$ .

In addition to the string winding number, the string oscillations will also contribute to the total momentum along the  $S^1$ . This total momentum is measured from the perspective of the five-brane world-volume, and is therefore related but not equal to the internal KK momentum. The relation between the two quantities is determined via a level matching condition equating this total string momentum to a zero-mode contribution. As before, the form of the zero-mode contribution is determined by means of the free field theory

of the massless five-brane modes, and thus depends quadratically on the fluxes and KK-momenta that gives rise to the electric charge  $q_e$ . This gives the second condition

$$\frac{1}{2}q_e^2 + \sum_{k,l,m,I} l N_{k,l,m}^I = 1. \quad (4.4)$$

as the constraint of translation invariance on the world-volume along the  $S^1$  direction.

Finally, since the D-brane intersection strings carry a non-zero space-time helicity  $m$ , we can also introduce a third level matching condition that equates the sum of the individual string contributions to the total helicity quantum number.

### *T-duality*

We can learn more about the dyonic five-brane configuration by considering the effect of the  $T$ -duality transformation that interchanges the electric and magnetic charges. Concretely, let us consider the configuration of a dyonic five-brane wound along the  $A$ -cycle of the internal two-torus  $T^2$ , and act on it with the simultaneous  $R \rightarrow 1/R$  transformation on both the  $A$ - and  $B$ -cycle. This map leaves the five-brane location invariant. However, when applied to the intersection strings within the five-brane, it changes their winding direction to the  $B$ -cycle, while in addition it interchanges the winding and momentum quantum numbers. It would appear therefore that the new intersection strings lie perpendicular to the five-brane.

An alternative conclusion, however, is that a dyonic five-brane should rather be thought of a pair of perpendicular five-branes, one around the  $A$ -cycle and one around the  $B$ -cycle. Both five-branes contain a collection of D-brane intersection strings, which however are  $T$ -duals of each other in that the momentum quantum numbers of the  $A$ -cycle strings are identified with the winding quantum numbers of the  $B$ -cycle strings. In this picture, the total winding number of the intersection strings around, say, the  $A$ -cycle is measured by  $q_m^2$ , while the total string winding number around the  $B$ -cycle is  $q_e^2$ .

### *A generating function*

It has been argued that the above multiple D-brane description leads to a two-dimensional sigma model with target space the sum of symmetric products of  $K3$  manifolds [7]. The 1/2 BPS states are in one-to-one correspondence with ground states of this model and correspond to harmonic forms. The number of these ground states follows from a direct application of the so-called orbifold formula, and is summarized in terms of

the generating function (see [19])

$$\sum_n e^{2\pi i n \sigma} \dim H^* \left( \frac{(K3)^n}{S_n} \right) = \prod_{l>0} \frac{1}{(1 - e^{2\pi i l \sigma})^{24}} \quad (4.5)$$

Our results now suggest a concrete generalization of this counting formula to the 1/4 BPS states, in which case one has to include the transversal string oscillations.

As was first suggested in [7], the counting of these states is naturally related to the elliptic genus of the above-mentioned sigma model. Based on our degeneracy formula, we conjecture that these elliptic genera have a generating function \*

$$\sum_n e^{2\pi i n \sigma} \chi_{\rho, v} \left( \frac{(K3)^n}{S_n} \right) = \prod_{k \geq 0, l > 0, m \in \mathbf{Z}} \frac{1}{(1 - e^{2\pi i (k\rho + l\sigma + mv)})^{c(4kl - m^2)}} \quad (4.6)$$

This conjectured identity suggests the following physical interpretation. The terms on the left-hand side count the oscillations of a single long string on the multiple product of the transversal target space  $K3$ . The right-hand side of our formula (4.6) on the other hand gives the usual second quantized representation of a multiple string state on  $K3$  in terms of a string field theory Fock space. The equality of both sides, if indeed true, indicates that the single long string state on the symmetric product of  $K3$  provides an exact first quantized representation of the appropriately symmetrized multiple string state on  $K3$ . Loosely speaking, we equate the elliptic genus of the ‘second-quantized manifold’ to the second-quantized elliptic genus of a single copy of the manifold. Based on this intuition, we expect that in the above relation the manifold  $K3$  can be replaced by an arbitrary manifold  $X$ , if one uses the expansion coefficients  $c(n)$  of the elliptic genus of  $X$ .

We note further that the above formula satisfies a number of highly non-trivial consistency checks. First, the  $n = 1$  term gives indeed the answer (3.6) for the  $K3$  elliptic genus. Secondly, for  $v = 0$  one recovers the ground state expression (4.5). Thirdly, the expression on the right-hand side of equation (4.6) has the required automorphic properties. Precisely because of that reason this particular expression has been considered before in the literature [23]. It has the important property that, when expanded in powers of  $e^{2\pi i \sigma}$  (the so-called Fourier-Jacobi expansion) the  $n$ -th coefficient is a weak Jacobi form of weight zero and index  $n$  for the modular group  $SL(2, \mathbf{Z})$  that acts on the modulus  $\rho$  and  $v$ . (See [27] for the theory of Jacobi forms.) The expansion coefficients are therefore indeed candidates for elliptic genera of Calabi-Yau manifolds of (complex) dimension  $2n$ , which matches the expansion on the left-hand side. Note that the space of weak Jacobi

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\*Here in the product we removed, relative to (3.8), the contribution of the states with  $l = 0$ . The resulting expression is exactly the contribution of the non-degenerate orbits in the fundamental domain integral [21].

forms of weight zero and fixed index is finite-dimensional. Given the fact that the Euler characters match, this gives extra support to our conjecture.

Finally, we note that the asymptotic growth of the right-hand side, as described in section 2, gives via the above equality a prediction for the number of single string states on  $(K3)^n/S_n$  at large oscillation level. This prediction can be verified by standard techniques.

## 5. CONCLUDING REMARKS

Let us finally mention some applications and possible generalizations of our result.

### *Five-dimensional degeneracies*

One of the consequences of our analysis is a concrete refinement of the results of Strominger and Vafa [7] on the counting of microscopic black hole states in five dimensions. The formula (4.6) gives an explicit counting formula of these states and in particular allows one to analyze the asymptotic growth in more directions by separately controlling the total string winding, momentum and helicity number. In five dimensions the level matching conditions equate these quantum numbers to respectively  $q^2$ , the charge  $Q_H$  of [7] and the five-dimensional spin  $J$ . Explicitly, the degeneracy of these states is given by

$$d(q, Q_H, J) = \oint dp d\sigma dv \frac{e^{i\pi(q^2\rho + Q_H\sigma + 2Jv)}}{\Phi'(\rho, \sigma, v)} \quad (5.1)$$

with  $\Phi'$  the denominator on the right-hand side of equation (4.6). The asymptotics of  $\Phi'$  are similar to those of  $\Phi$  as analyzed in the Appendix.

### *Generalizations*

It would be interesting to extend our results to more general four-dimensional black hole states, by allowing for the possibility of non-zero angular momentum [20] or states away from extremality.

Spinning extremal black holes are in fact most naturally included in our description, although for this we would need to give up manifest  $SL(2, \mathbf{Z})$ -duality. The most general spinning dyonic black hole geometry is characterized by *four* independent macroscopic quantities (namely the angular momentum  $J$ , the length-squared of the charges  $q_e^2$ ,  $q_m^2$ , and  $q_e \cdot q_m$ ). Our most general states, on the other hand, carry at most three macroscopic quantum numbers. In our presentation we eventually arrived at a duality symmetric expression by equating the total ‘internal’ helicity to the ‘external’ helicity  $q_e \cdot q_m$ . However,



we can in principle omit this constraint and simply identify the total internal helicity with the macroscopic external angular momentum. This will give a degeneracy formula consistent with the predicted entropy for spinning black holes [20] in the case that  $q_e \cdot q_m = 0$ .

Another important generalization is to include near extremal black hole states [28]. Within our framework, an obvious step is to relax the BPS condition on the individual strings, and allow for arbitrary left- and right-moving oscillations. We expect that a naive extension of our formalism in this regime would indeed produce an asymptotic number of states in accordance with the four-dimensional macroscopic entropy proportional to the mass squared. This is in essence a consequence of the fact that the mass-shell relation  $L_0 = (\text{mass})^2$  is applied twice in our formalism: once on the string on the world-volume, and once on the five-brane in space-time. By a similar argument we also expect that the energy gap will also behave in accordance with the thermodynamic prediction. We should stress however that the resulting description will very likely be of limited validity, since on various crucial points we have made use of the BPS condition. In particular it is not at all clear that the (effective) string theory on the world-volume can still be treated as an (approximately) free string theory.

### *Algebraic structure*

Our results suggest that there exists an interesting algebraic structure behind the  $N = 4$  dyon spectrum, related to a generalized Kac-Moody (GKM) algebra. The particular (super)algebra that underlies our degeneracy formula is a so-called automorphic correction [23] of a hyperbolic Kac-Moody algebra with three simple roots with Cartan matrix

$$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix} \quad (5.2)$$

(Note that the corresponding GKM has an infinite number of imaginary simple roots, roughly corresponding to the single string BPS states.) The fact that the denominator of this GKM appears (in the denominator) of our partition function, suggests that perhaps the full characters, including the numerator, also play a role in the BPS spectrum. Perhaps it is possible to realize the spectrum generating algebra of the  $N = 4$  dyon BPS states in terms of this symmetry algebra (cf. [29]). Generalized Kac-Moody algebras also recently appeared in the  $N = 2$  context [22], where it was suggested that the heterotic string vertex operators representing the perturbative BPS states form a GKM.

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## APPENDIX

The asymptotic behavior of the degeneracy formula (2.3) are related to the zeroes of the automorphic form  $\Phi$  in the integrand. These zeroes are well understood using either standard facts on genus two Riemann surfaces, the results of Borcherds [10] or the representation in terms of a string one-loop amplitude [22, 21] as in (3.5). From the latter point of view we understand that  $\Phi(\rho, \sigma, v)$  acquires a zero or pole whenever the moduli  $\rho$ ,  $\sigma$  and  $v$  that parametrize the Narain lattice  $\Gamma^{3,2}$  allow chiral vertex operators, *i.e.* elements  $p = (p_L, p_R) \in \Gamma^{3,2}$  with  $p_R = 0$ . In terms of the integer lattice vector  $p = (k, l, m, a, c) \in \Gamma^{3,2} \cong \mathbf{Z}^5$  we have [21]

$$p_R^2 = \frac{1}{2Y} |k\rho + l\sigma + mv + a(\rho\sigma - v^2) + c|^2 \quad (\text{A.1})$$

with  $Y = \rho_2\sigma_2 - v_2^2$ , and

$$p^2 = \frac{1}{2}m^2 - 2kl + 2ac. \quad (\text{A.2})$$

If we use the vector notation  $y = (\rho, \sigma, 2v)$ , with  $-\frac{1}{2}y^2 = \rho\sigma - v^2$ , we see that the zeroes and poles occur at the ‘rational quadratic divisors’ of [10]

$$-\frac{1}{2}ay^2 + b \cdot y + c = 0 \quad (\text{A.3})$$

where  $b = (k, l, m) \in \Gamma^{2,1}$  with norm-squared  $b^2 = -2kl + \frac{1}{2}m^2$  and  $a, c \in \mathbf{Z}$ . We can write equation (A.3) also as  $p \cdot v = 0$ , where we have introduced the 5-dimensional null vector  $v = (y; -\frac{1}{2}y^2, 1)$ . Of course  $|p \cdot v|^2 \sim p_R^2$ . In terms of the coefficients  $c(n)$ , the order of the zero of  $\Phi(y)$  at such a divisor is  $\sum_{n>0} c(-2n^2p^2)$  with  $p^2 > 0$ . In our case the only possibility is  $c(-1) = 2$ , so we necessarily have  $p^2 = 1/2$  and the partition function  $1/\Phi(y)$  has a second order pole at this divisor.

In the theory of genus two curves or abelian surfaces, the equation (A.3) is well-known to describe (a component of) the so-called Humbert surface  $H_\Delta$  with discriminant  $\Delta = 2p^2$  [30]. In our case we are interested in the surface  $H_1$ . This is indeed known to be the zero-locus of the product of all even theta-functions.

The ‘tachyon pole’ at  $v = 0$  occurs for  $b = (0, 0, 1)$  and  $a = c = 0$ , which describes the factorization on a product of two elliptic curves. All the other poles are transformations of this one by  $Sp(2, \mathbf{Z}) \cong SO(3, 2, \mathbf{Z})$ . This group acts linear on the vectors  $v$  and  $p$ , which is another way to see that the divisors are of the form (A.3). The leading contribution in the Fourier coefficient

$$D(q) = \int d^3y \frac{e^{2\pi i q \cdot y}}{\Phi(y)} \quad (\text{A.4})$$

for large  $q$  from the divisor (A.3) is of the form  $D(q) \sim e^{2\pi\mu|q|}$  with  $\mu^2 = (b^2 + 2ac)/a^2 = 1/2a^2$ . So the dominant term will be given by  $a = 1$ . This gives indeed the black-hole entropy (2.9). The other coefficients are then constrained by  $4c - 4kl + m^2 = 1$ , so in particular  $m$  is odd, and  $k, l$  can be arbitrary. However, since we have the freedom to shift the variables  $\rho, \sigma, v$  by integers, we can put  $k, l, c = 0$  and  $m = 1$ . The divisor is then given by  $\rho\sigma - v^2 + v = 0$ .

Let us finally make a few remarks about the geometrical interpretation in terms of a genus two surface. Let  $A_1, A_2, B_1, B_2$  be a basis for the homology group  $H_1(\Sigma)$  of a genus two Riemann surface  $\Sigma$ . Now consider the quotient of  $\Lambda^2 H_1$  by the relation

$$A_1 \wedge B_1 + A_2 \wedge B_2 = 0 \quad (\text{A.5})$$

and call the resulting 5-dimensional lattice  $L \cong \Gamma^{3,2}$ . It naturally carries a signature  $(3, 2)$  bilinear form, coming from the intersection form on  $\Lambda^2 H_1$ . This makes explicit the isomorphism  $Sp(2, \mathbf{Z}) \cong SO(3, 2, \mathbf{Z})$ . As a basis of  $L$  we choose

$$B_1 \wedge A_2, B_2 \wedge A_1, A_1 \wedge B_1 = B_2 \wedge A_2, B_1 \wedge B_2, A_1 \wedge A_2. \quad (\text{A.6})$$

There is a natural map  $L \rightarrow \mathbf{C}$  given in terms of the periods of the abelian differentials  $\omega_1, \omega_2$

$$C \wedge D \rightarrow \oint_C \omega_1 \oint_D \omega_2 - \oint_D \omega_1 \oint_C \omega_2 \quad (\text{A.7})$$

Under this map our basis (A.6) gives the ‘bi-periods’

$$\rho, \sigma, v, \rho\sigma - v^2, 1. \quad (\text{A.8})$$

It is clear that symplectic transformations will act linear on these variables. To any basis element of  $L$  we can associate a zero-homology cycle in the homotopy group  $\pi_1(\Sigma)$  by

$$C \wedge D \rightarrow [C, D] = CDC^{-1}D^{-1} \quad (\text{A.9})$$

Note that  $[A_1, B_1][A_2, B_2] \sim 1$  so that this map is well-defined. It associates a bi-period to a vanishing cycle. Indeed, in this way the divisor  $v = 0$  describes the usual pinching along the dividing cycle  $[A_1, B_1] = [B_2, A_2]$ . Other divisors can be similarly interpreted.

## References

- [1] A. Font, L.E. Ibanez, D. Lust, and F. Quevedo, “Strong-Weak Coupling Duality and Nonperturbative Effects in String Theory,” *Phys.Lett.* **B249** (1990) 35-43.
- [2] A. Sen, “Strong-Weak Coupling Duality in Four Dimensional String Theory,” *Int.J.Mod.Phys.* **A9** (1994) 3707-3750, hep-th/9402002; “Dyon-Monopole Bound States, Self-Dual Harmonic Forms on the Multi-Monopole Moduli Space, and  $SL(2, Z)$  Invariance in String Theory, *Phys.Lett.* **B329** (1994) 217-221, hep-th/9402032.
- [3] J.P. Gauntlett and J.A. Harvey, “ $S$ -Duality and the Spectrum of Magnetic Monopoles in Heterotic String Theory,” hep-th/9407111.
- [4] J. Bekenstein, *Phys. Rev.* **D7** (1973) 2333; *Phys. Rev.* **D9** (1974) 3292.  
S. W. Hawking, *Phys. Rev.* **D13** (1976) 191.
- [5] M. Cvetič and D. Youm, “Dyonic BPS Saturated Black Holes in Heterotic String on a Six Torus,” *Phys.Rev.* **D53** (1996) 584-588; hep-th/9507090. M. Cvetič and A. Tseytlin, “Solitonic Strings and BPS Saturated Dyonic Black Holes,” hep-th/9512031.
- [6] J. Polchinski, “Dirichlet-Branes and Ramond-Ramond Charges,” hep-th/9510017; J. Polchinski, S. Chaudhuri, and C. Johnson, “Notes on D-Branes,” hep-th/9602052.
- [7] A. Strominger, C. Vafa, “Microscopic Origin of the Bekenstein-Hawking Entropy,” hep-th/9601029.
- [8] F. Larsen and F. Wilczek, “Internal Structure of Black Holes,” hep-th/9511064; M. Cvetič and A. Tseytlin, “Solitonic Strings and BPS Saturated Dyonic Black Holes,” hep-th/9512031; G. Horowitz and A. Strominger, “Counting States of Near-Extremal Black Holes,” hep-th/9602051; C. Callan and J. Maldacena, “D-brane Approach to Black Hole Quantum Mechanics,” hep-th/9602043.
- [9] C. Johnson, R. Khuri and R. Myers, “Entropy of 4D Extremal Black Holes,” hep-th/9603061; J. Maldacena and A. Strominger, “Statistical Entropy of Four-Dimensional Extremal Black Holes,” hep-th/9603060.
- [10] R. E. Borcherds, “Automorphic Forms on  $O_{s+2,2}(R)$  and Infinite Products,” *Invent. Math.* **120** (1995) 161.
- [11] G. Moore, “Modular forms and two-loop string physics,” *Phys.Lett.* **176B** (1986) 369; A. Belavin, V.Knizhnik, A. Morozov and A. Perelemov, *Phys. Lett.* **B 177** (1986) 324.
- [12] J. A. Harvey and A. Strominger, “The Heterotic String is a Soliton,” *Nucl. Phys.* **B 449** (1995) 535-552; *Nucl. Phys.* **B458** (1996) 456-473, hep-th/9504047.

- [13] M.J. Duff, R.R. Khuri, and J.X. Lu, “String Solitons,” Phys.Rept. **259** (1995) 213-326, hep-th/9412184. C. Callan, J. Harvey, and A. Strominger, Nucl.Phys. **B367** (1991) 60.
- [14] R. Dijkgraaf, E. Verlinde and H. Verlinde, “BPS Quantization of the Five-Brane,” hep-th/9604055.
- [15] R. Dijkgraaf, E. Verlinde and H. Verlinde, “BPS Spectrum of the Five-Brane and Black Hole Entropy,” hep-th/9603126.
- [16] L. Dixon, V. Kaplunovsky, and J. Louis, “Moduli-dependence of string loop corrections to gauge coupling constants,” Nucl. Phys. **B307** (1988) 145.
- [17] E. Verlinde, “Global Aspects of Electric-Magnetic Duality,” Nucl.Phys. **B455** (1995) 211-228, hep-th/9506011
- [18] T. Kawai, Y. Yamada and S.-K. Yang, “Elliptic Genera and N=2 Superconformal Field Theory,” Nucl. Phys. **B414** (1994) 191-212. T. Eguchi, H. Ooguri, A. Taormina, S.-K. Yang, “Superconformal Algebras and String Compactification on Manifolds with  $SU(N)$  Holonomy,” Nucl.Phys. **B315** (1989) 193.
- [19] C. Vafa and E. Witten, “A Strong Coupling Test of S-Duality,” Nucl.Phys. **B431** (1994) 3-77, hep-th/9408074.
- [20] J.C. Breckenridge, R.C. Myers, A.W. Peet and C. Vafa, “D-branes and Spinning Black Holes,” hep-th/9602065; J.C. Breckenridge, D.A. Lowe, R.C. Myers, A.W. Peet, A. Strominger and C. Vafa, “Macroscopic and Microscopic Entropy of Near-Extremal Spinning Black Holes,” hep-th/9603078.
- [21] T. Kawai, “ $N = 2$  Heterotic String Threshold Correction,  $K3$  Surface and Generalized Kac-Moody Superalgebra,” hep-th/9512046.
- [22] J. Harvey and G. Moore, “Algebras, BPS States, and Strings,” Nucl.Phys.**B 463** (1996) 315-368
- [23] V.A. Gritsenko and V.V. Nikulin, “Siegel Automorphic Form Corrections of Some Lorentzian Kac-Moody Algebras,” alg-geom/9504006.
- [24] P. Mayr and S. Stieberger, Phys. Lett. **B 355** (1995) 107, hep-th/9504129.
- [25] C. Vafa, “Gas of D-Branes and Hagedorn Density of BPS States,” hep-th/9511026, “Instantons on D-branes,” hep-th/9512078; M. Bershadsky, V. Sadov, and C. Vafa, “D-Branes and Topological Field Theories,” hep-th/9511222.
- [26] A. Sen, “A Note on Marginally Stable Bound States in Type II String Theory” hep-th/9510229; “U Duality and Intersecting D-Branes,” hep-th/9511026; “T-Duality of p-Branes,” hep-th/9512062.
- [27] M. Eichler and D. Zagier, “The Theory of Jacobi Forms” (Birkhäuser, 1985).

- [28] G. Horowitz and A. Strominger, “Counting States of Near-Extremal Black Holes,” hep-th/9602051; I. R. Klebanov, A. A. Tseytlin, “Entropy of Near-Extremal Black p-branes,” hep-th/9604089; J. M. Maldacena, “Statistical Entropy of Near Extremal Five-branes,” hep-th/9605016.
- [29] A. Giveon and M. Porrati, “Duality Invariant String Algebra and  $D = 4$  Effective Actions,” Nucl.Phys. **B355** (1991) 422-454.
- [30] G. van der Geer, “Hilbert Modular Surfaces” (Springer, 1980).